UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3401**

ASSESSMENT PATTERN : **MATH3401A**

MODULE NAME : **Methods of Mathematical Physics I**

DATE : **20-May-09**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2008/09-MATH3401A-001-EXAM-12

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Use the method of variation of parameters to obtain a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = e^{4x}\cosh(x).$$

(b) Determine a function $f(t)$ of the complex variable t and contours C_1 and C_2 in the complex t-plane so that the functions

$$y_n(x) = \int_{C_n} e^{xt} f(t)dt, \qquad n = 1, 2,$$

are non-trivial, linearly independent solutions of the differential equation

$$x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} - (8x+4)y = 0, \quad x > 0.$$

Find the contours C_1 and C_2 such that $y_1(x)$ is found in closed form and is bounded at $x = 0$, and $y_2(x)$ is unbounded at $x = 0$.

2. (a) Suppose that the constants A, B, C, D are such that $(x, y) = (0, 0)$ is a singular point of the phase equation

$$\frac{dy}{dx} = \frac{Cx + Dy}{Ax + By}.$$

Describe four types of singularity, depending on A, B, C, D, and explain under which conditions the singular point is stable or unstable.

(b) For a dynamical system, show that periodic motion occurs if and only if a corresponding phase trajectory is closed.

(c) Consider the equation

$$\ddot{x} = x(x^3 - \lambda^3),$$

where a dot represents differentiation with respect to time t and $\lambda > 0$ is a constant. Find all singular points of the system, determine their nature and sketch the trajectories in the phase plane.

If $y = U$ at $x = 0$, show that periodic motion can occur only if

$$U^2 < 3\frac{\lambda^5}{5}.$$

Write down a formula for the period of the motion for a closed phase trajectory γ.

3. Let $x = x(t)$ satisfy the equation

$$\ddot{x} + \epsilon\dot{x}(4x^2 - 1) + x = 0,$$

where a dot represents differentiation with respect to the variable t. Considering a *periodic* solution $x(t)$ with period T and frequency $n = 2\pi/T$, show that the above equation can be written as

$$n^2\frac{d^2x}{d\theta^2} + \epsilon n\frac{dx}{d\theta}(4x^2 - 1) + x = 0$$

by making the substitution $\theta = nt$. Explain why $x(\theta)$ is therefore 2π-periodic. Assuming that $0 < \epsilon \ll 1$, seek solutions of the form

$$x(\theta) = x_0(\theta) + \epsilon x_1(\theta) + \epsilon^2 x_2(\theta) + \dots\,,$$

$$n = n_0 + \epsilon n_1 + \epsilon^2 n_2 + \dots\,.$$

If $x(0) = 0$ show that

$$x_0 = \sin(\theta), \quad x_1 = \frac{1}{8}(\cos(\theta) - \cos(3\theta)) + B_1\sin(\theta),$$

where B_1 is an unknown constant. Also show that $n_0 = 1$ and $n_1 = 0$. Derive the differential equation for $x_2(\theta)$.

HINT: $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$.

4. In this question a dot represents differentiation with respect to the variable t; $f(x)$ is an integrable function of x; and ϵ is a constant. Consider the differential equation

$$\ddot{x} + \epsilon f(x)\dot{x} + x = 0, \quad \epsilon > 0.$$

(a) By applying the Lienard transformation, show that the phase equation for the above system has the form

$$\frac{dy}{dx} = \frac{x}{\epsilon F - y},$$

with $F'(x) = f(x)$.

(b) Assuming that for each ϵ a periodic solution $x(t)$ exists, show that for the corresponding closed phase trajectory γ in the Lienard plane,

$$\oint_\gamma y f(x) dx = 0$$

holds true.

(c) For $f(x) = 2x^2 - 8$ sketch the periodic trajectory in the Lienard phase plane assuming that $\epsilon \gg 1$. Show that the period T is given by

$$T = \epsilon(24 - 16 \ln 2)$$

to leading order in ϵ. Also show that the amplitude A is given by $A = 4$ to leading order in ϵ.

HINT: The cubic $2x^3/3 - 8x - 32/3$ has roots at $x = -2$ and 4 and the cubic $2x^3/3 - 8x + 32/3$ has roots at $x = -4$ and 2.

5. (a) State *without proof* one of the two forms of Watson's Lemma.

(b) Throughout the interval $a \leq t \leq b$ the function $f(t)$ is continuous and the function $\phi(t)$ has a maximum at $t = t_0$ with $a < t_0 < b$ and $\phi''(t_0) \neq 0$. Show that as $x \to +\infty$

$$I_1(x) = \int_a^b e^{x\phi(t)} f(t) dt \sim e^{x\phi(t_0)} f(t_0) \left(-\frac{2\pi}{\phi''(t_0)x} \right)^{\frac{1}{2}}.$$

State without proof how this result changes if $t_0 = a$ or $t_0 = b$?

(c) Use the method of stationary phase to find an asymptotic form for

$$I_2(x) = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{ix\cos(t)} dt$$

valid as $x \to +\infty$. Hence deduce that

$$\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(x \cos(t)) dt \sim 2 \cos\left(x - \frac{\pi}{4}\right) \left(\frac{2\pi}{x}\right)^{\frac{1}{2}}$$

as $x \to +\infty$.